Variational calculation of the effective fluid permeability of heterogeneous media

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We evaluate the effective permeability of heterogeneous media with Gaussian local permeability disorder using the replica-variational approach. We obtain integral equations that determine the effective permeability kernel, and we study specific cases that admit analytical solutions. Specifically, in the case of homogeneous disorder we obtain a variational estimate for the uniform effective permeability. We compare the results of our analytical calculations with experimental and numerical data. Finally, we model the behavior of the effective permeability in the preasymptotic regime by means of momentum filters. Explicit finite-size expressions are obtained in terms of a support function that increases monotonically with the ratio of the support scale over the correlation length of the disorder. It is found that the asymptotic effective permeability is approached at a slower rate than expected. [S1063-651X(97)10806-6]

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I. INTRODUCTION

Heterogeneous media have important technological and environmental applications, and their properties over a wide range of scales are a subject of continuing investigation. In particular, the transport properties of geological porous media are crucial in determining the fate of pollutants dispersed in the ground water [1] as well as for oil recovery [2]. The complexities of heterogeneous media are due to the complex structure of the fluid-solid interface and partial characterization. In view of these considerations, random fields [3] are often used to represent heterogeneous media at a certain local scale. The properties of the medium at larger scales are obtained by means of effective parameters that involve averaging over the random fluctuations. Within the stochastic framework, the effective permeability is defined as the linear coefficient that relates the mean fluid flux to the mean pressure gradient. However, this definition is not unique: while it is commonly used in analytical studies [4,5], in numerical simulations a definition based on the spatial average (sample mean) of the local flux is preferred [6]. The stochastic and sample means are equal if and only if the flux and the pressure gradient are ergodic. Since the ergodicity is a thermodynamic limit property [7], a necessary (but not sufficient) condition for using the sample mean as an estimator of the stochastic mean is that the support size be infinitely larger than the correlation range of the disorder fluctuations; this condition is not met in the case of long-range correlations. In the case of short-range correlations, the ratio of the linear size of the support over the correlation length represents the dimensionless support scale.

Random fields are used to represent the correlations of the porous medium. Hence, we do not explicitly investigate critical scaling near the percolation threshold [8]. In the case of homogeneous disorder the correlation functions are translationally invariant, thus leading to a uniform effective permeability. In the asymptotic limit, which is obtained for infinite dimensionless support scale, the effective permeability becomes independent of the support scale [1,9]. The preasymptotic regime is dispersive, and the effective permeability is represented by means of a non-local effective kernel [10–13]. In the case of homogeneous mean pressure gradient a uniform effective permeability is obtained that depends on the support scale [14,15] and the boundary conditions [16]. We were motivated to study the preasymptotic behavior in light of experimental studies that show a definite scale dependence. Field measurements exhibit [16,17] an increase of the permeability with the support scale. On the other hand, small-scale laboratory experiments on natural porous media and numerical investigations indicate that the effective permeability decreases with increasing support scale [18].

We use the replica-variational approach of [19,20] in order to evaluate the ensemble average over the fluctuations. This approach leads to a system of integral equations for the effective permeability kernel that we solve explicitly in specific cases. In the case of homogeneous disorder the nonlocal kernel is shown to lead to the well-known uniform effective permeability. Our theoretical estimate is shown to approximate accurately experimental measurements of the effective permeability in sandstone blocks. The effects of finite support scale are modeled by means of momentum filters that cut off the nonlocal kernels at a threshold determined by the scale of the support.

This paper is structured as follows: In Sec. II we present the stochastic formulation of flow, and we derive the integral equations that determine the effective permeability kernel. In Sec. III we obtain general expressions for the kernel as well as estimates of the asymptotic effective permeability for both isotropic and anisotropic disorder. In Sec. IV we derive by perturbation analysis an explicit expression for the dispersive part of the kernel. In Sec. V we obtain approximate estimates for the finite-scale behavior of the effective permeability by means of momentum filters. Finally, in Sec. VI we compare our analytical estimates with experimental and numerical results.

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II. STOCHASTIC FORMULATION OF FLOW IN HETEROGENEOUS MEDIA

In the following, we use the Einstein summation convention for vector and tensor indices. An unpaired index indicates that the expression be repeated for all values of the index from 1 to d, where d denotes the dimensionality of the medium. The tilde denotes the Fourier transform. The overbar denotes ensemble averaging over the quenched Gaussian disorder. The angular brackets denote the statistical average with respect to the probability distribution of the auxiliary fields. The trace operator is denoted by Tr.

Consider a heterogeneous medium with a random permeability represented locally by the random field $K(\mathbf{s})$, which is assumed to be locally isotropic and macroscopically homogeneous, namely,

$$K(\mathbf{s})=\overline{K}+\delta K(\mathbf{s}), \tag{2.1}$$

where $\overline{K}$ denotes the arithmetic mean and $\delta K(\mathbf{s})$ the zero-mean fluctuations. We assume that the fluctuations are normally distributed with a two-point correlation function given by

$$c_K(\mathbf{s}-\mathbf{s}')=\overline{\delta K(\mathbf{s})\,\delta K(\mathbf{s}')}. \tag{2.2}$$

The variance of the permeability fluctuations is denoted by $\sigma_K^2=c_K(0)$ and the coefficient of variation by $\mu=\sigma_K/\overline{K}$. The local pressure is denoted by $P(\mathbf{s})$ and its gradient by $\mathbf{J}(\mathbf{s})=-\boldsymbol{\nabla}P(\mathbf{s})$. The fluid flux is determined from Darcy's law $\mathbf{Q}(\mathbf{s})=K(\mathbf{s})\mathbf{J}(\mathbf{s})$. The equation of flux continuity for steady-state flow is given by $\boldsymbol{\nabla}\cdot\mathbf{Q}(\mathbf{s})=0$, and in view of Eq. (2.1) it can be expressed as follows:

$$\overline{K}\nabla^2P(\mathbf{s})+\boldsymbol{\nabla}\cdot\delta K(\mathbf{s})\boldsymbol{\nabla}P(\mathbf{s})=0. \tag{2.3}$$

In a homogeneous medium of uniform permeability $\overline{K}$ the pressure satisfies the Laplace equation $\nabla^2P_0(\mathbf{s})=0$. The associated Green's function satisfies the Poisson equation

$$\overline{K}\nabla^2G_0(\mathbf{s},\mathbf{s}')=-\delta(\mathbf{s}-\mathbf{s}'). \tag{2.4}$$

The solution of Eq. (2.3) can be expressed in a self-consistent form as follows:

$$J_i(\mathbf{s})=J_{0i}(\mathbf{s})+\int d\mathbf{s}'G_{0ij}(\mathbf{s},\mathbf{s}')\,\delta K(\mathbf{s}')J_j(\mathbf{s}'), \tag{2.5}$$

where $\mathbf{J}_0(\mathbf{s})=-\boldsymbol{\nabla}P_0(\mathbf{s})$ is the deterministic component of the pressure gradient and $G_{0ij}(\mathbf{s},\mathbf{s}')$ is the dipolar tensor [21]

$$G_{0ij}(\mathbf{s},\mathbf{s}')=\frac{\partial^2G_0(\mathbf{s},\mathbf{s}')}{\partial s_i\partial s_j}. \tag{2.6}$$

By means of an iterative expansion of the pressure gradient in Eq. (2.5) the Neumann series is generated. Summation of the series leads to the following exact expression

$$J_i(\mathbf{s})=\int \mathbf{ds}'\,T_{ij}^{-1}(\mathbf{s},\mathbf{s}')J_{0j}(\mathbf{s}'), \tag{2.7}$$

where the tensor kernel T_{ij} is given by

$$T_{ij}(\mathbf{s},\mathbf{s}')=\delta_{ij}\,\delta(\mathbf{s}-\mathbf{s}')-G_{0ij}(\mathbf{s},\mathbf{s}')\,\delta K(\mathbf{s}'). \tag{2.8}$$

In the case of constant $\overline{\mathbf{J}}$ and homogeneous disorder, the mean flux $\overline{\mathbf{Q}}$ is constant, and it leads to the local effective Darcy's law $\overline{\mathbf{Q}}=\mathbf{K}_{\rm eff}\overline{\mathbf{J}}$, where $\mathbf{K}_{\rm eff}$ is the uniform effective permeability [1]. In the dispersive case the mean flux is given by means of the nonlocal expression

$$\overline{Q_i(\mathbf{s})}=\int \mathbf{ds}'\;K_{ij}^*(\mathbf{s},\mathbf{s}')\overline{J_j(\mathbf{s}')}, \tag{2.9}$$

where $K_{ij}^*(\mathbf{s},\mathbf{s}')$ is the effective permeability kernel. In light of Eqs. (2.5) and (2.9) the mean pressure gradient is related to the effective permeability kernel via the integral equation

$$\overline{J_i(\mathbf{s})}=J_{0i}(\mathbf{s})-\int \mathbf{ds}'\int \mathbf{ds}''G_{0ij}(\mathbf{s},\mathbf{s}')\,\delta K_{jm}^*(\mathbf{s}',\mathbf{s}'')\overline{J_m(\mathbf{s}'')}, \tag{2.10}$$

where $\delta K_{jm}^*(\mathbf{s},\mathbf{s}')=\overline{K}\delta_{jm}\delta(\mathbf{s}-\mathbf{s}')-K_{jm}^*(\mathbf{s},\mathbf{s}')$ represents the fluctuation component of the kernel. An equivalent expression can be obtained by means of Eq. (2.7), i.e.,

$$\overline{J_i(\mathbf{s})}=\int \mathbf{ds}'M_{ij}(\mathbf{s},\mathbf{s}')J_{0j}(\mathbf{s}'), \tag{2.11}$$

where

$$M_{ij}(\mathbf{s},\mathbf{s}')=\overline{T_{ij}^{-1}(\mathbf{s},\mathbf{s}')}. \tag{2.12}$$

In view of Eqs. (2.10) and (2.11) the following relationship is established between the permeability kernel and the two-point correlation function $M_{ij}(\mathbf{s},\mathbf{s}')$:

$$\widetilde{M}_{ij}^{-1}(\mathbf{k})=\delta_{ij}+\widetilde{G}_{0ik}(\mathbf{k})\,\delta\widetilde{K}_{kj}^*(\mathbf{k}), \tag{2.13}$$

where

$$\delta\widetilde{K}_{kj}^*(\mathbf{k})=\overline{K}\delta_{kj}-\widetilde{K}_{kj}^*(\mathbf{k}), \tag{2.14}$$

The correlation function $\widetilde{M}_{ij}(\mathbf{k})$ can be calculated using the field-theoretic formulation of [20] that we outline briefly here. First, the field $T_{ij}^{-1}(\mathbf{s},\mathbf{s}')$ is expressed in terms of d auxiliary fields $\phi_i(\mathbf{s})$, $i=1,\ldots,d$ and their complex conjugates $\phi_i^*(\mathbf{s})$ as follows:

$$T_{ij}^{-1}(\mathbf{s},\mathbf{s}')=\langle\phi_i^*(\mathbf{s})\,\phi_j(\mathbf{s}')\rangle. \tag{2.15}$$

The brackets denote the average with respect to the Gaussian measure

$$P(\phi_i^*,\phi_i)=\frac{1}{Z}\exp\left(-\sum_{i,j=1}^{d}\int \mathbf{ds}\int \mathbf{ds}'\,\phi_i^*(\mathbf{s})\times T_{ij}(\mathbf{s},\mathbf{s}')\,\phi_j(s')\right), \tag{2.16}$$

where the integral $\Sigma_{i,j=1}^{d}\int\mathbf{ds}\int\mathbf{ds}'\,\phi_i^*(\mathbf{s})T_{ij}(\mathbf{s},\mathbf{s}')\,\phi_j(\mathbf{s}')$ represents the disorder Hamiltonian and Z denotes the corresponding partition function. Using the replica trick [22], i.e., $Z^{-1}\overline{\ln Z}=\lim_{n\to 0}(\ln\overline{Z^n})/n$, we obtain the following expression for the correlation function (2.15):

$$\overline{T_{ij}^{-1}(\mathbf{s},\mathbf{s}')}=\lim_{n\to 0}\left\langle \sum_{\alpha=1}^{n} \frac{\phi_i^{*\alpha}(\mathbf{s})\,\phi_j^{\alpha}(\mathbf{s}')}{n}\right\rangle_{\mathcal{H}_{\rm eff}}, \tag{2.17}$$

where $\mathcal{H}_{\rm eff}$ denotes the effective Hamiltonian and $\langle\ \rangle_{\mathcal{H}_{\rm eff}}$ the average with respect to the effective measure, which involves a functional integral of the replica fields $\phi_i^{\alpha}(\mathbf{s})$, $\phi_i^{*\alpha}(\mathbf{s})$, where $i=1,\ldots,d$ and $\alpha=1,\ldots,n$. The standard procedure for evaluating the stochastic average of $T_{ij}^{-1}(\mathbf{s},\mathbf{s}')$ over the disorder is as follows: first, the disorder average of the expression (2.17) is determined assuming that the number of replicas is an integer; this leads to an effective probability measure. Then, the functional integral over the replicas is estimated using a variational approximation. The latter is obtained by establishing an upper bound for the free energy functional of the effective probability measure by means of a variational free energy. Finally, the correlation function $M_{ij}(\mathbf{s},\mathbf{s}')$ is obtained by minimizing the upper bound at the limit $n\to 0$. This procedure is based on the assumptions that the order of the zero-replica limit and the stochastic average operator can be commuted, and that the analytic continuation of the average over the replica fields when $n\to 0$ exists. The first step leads to a disordered-averaged probability measure $\overline{P(\phi_i^{\alpha},\phi_i^{*\alpha})}\propto\exp[-\mathcal{H}_{\rm eff}(\phi_i^{\alpha},\phi_i^{*\alpha})]$. The effective Hamiltonian $\mathcal{H}_{\rm eff}(\phi_i^{\alpha},\phi_i^{*\alpha})$ is given by

$$\mathcal{H}_{\rm eff}(\phi_i^{\alpha},\phi_i^{*\alpha})=\sum_{i=1}^{d}\sum_{\alpha=1}^{n}\int \mathbf{ds}|\phi_i^{\alpha}(\mathbf{s})|^2-\tfrac{1}{2}\sum_{i,j,k,l=1}^{d}\sum_{\alpha,\beta=1}^{n}\int \mathbf{ds}\int \mathbf{ds}'\int \mathbf{ds}_1\int \mathbf{ds}_2 c_K(\mathbf{s}_1-\mathbf{s}_2)G_{0ij}(\mathbf{s}-\mathbf{s}_1)$$
$$\times G_{0kl}(\mathbf{s}'-\mathbf{s}_2)\,\phi_i^{*\alpha}(\mathbf{s})\,\phi_j^{\alpha}(\mathbf{s}_1)\,\phi_k^{*\beta}(\mathbf{s}')\,\phi_l^{\beta}(\mathbf{s}_2) \tag{2.18}$$

and it involves a standard quartic coupling between replica fields. The second step involves the approximation of the effective probability measure by means of a variational Gaussian measure

$$P_0(\phi_i^{\alpha},\phi_i^{*\alpha})\propto\exp[-\mathcal{H}_0(\phi_i^{\alpha},\phi_i^{*\alpha})], \tag{2.19}$$

where the Hamiltonian $\mathcal{H}_0(\phi_i^{\alpha},\phi_i^{*\alpha})$ couples the replica fields diagonally by means of the two-point function $C_{ij}^{-1}(\mathbf{s},\mathbf{s}')$. The variational approximation of the correlation function $M_{ij}(\mathbf{s},\mathbf{s}')$ is given by the optimal variational correlation, denoted by $C_{ij}^{*}(\mathbf{s},\mathbf{s}')$. The latter is obtained using the following variational inequality

$$f\leq f_0+\lim_{n\to 0}\frac{\langle \mathcal{H}_{\rm eff}(\phi_i^{\alpha},\phi_i^{*\alpha})-\mathcal{H}_0(\phi_i^{\alpha},\phi_i^{*\alpha})\rangle_0}{Vn}, \tag{2.20}$$

where V denotes the volume of the system, f denotes the free-energy density (per unit volume and per replica) functional of the effective probability measure, f_0 denotes the Gaussian free energy density, and $\langle\ \rangle_0$ denotes the average over the Gaussian measure. Thus, the replica averages on the right-hand side of Eq. (2.20) can be explicitly evaluated by means of Gaussian integrals. The final step involves the minimization of the functional on the right-hand side of Eq. (2.20) with respect to the variational correlation. This procedure, in view of Eq. (2.13), leads to the following relation for the effective permeability kernel:

$$\widetilde{K}_{kj}^{*}(\mathbf{k})-\delta_{kj}\overline{K}=\int\frac{\mathbf{dk}_1}{(2\pi)^d}\,\widetilde{c}_K(\mathbf{k}_1)\widetilde{M}_{km}(\mathbf{k}-\mathbf{k}_1)$$
$$\times\widetilde{G}_{0mj}(\mathbf{k}-\mathbf{k}_1), \tag{2.21}$$

where in Eq. (2.21) $\widetilde{C}_{ij}^{*}(\mathbf{k})$ has been replaced with $\widetilde{M}_{ij}(\mathbf{k})$. The expressions (2.13) and (2.21) provide a system of equations that determines the effective permeability kernel $\widetilde{K}_{kj}^{*}(\mathbf{k})$. The latter includes the mean-field contribution $\overline{K}\mathbf{I}$, and the fluctuation part $\delta\widetilde{K}_{kj}^{*}(\mathbf{k})$ that represents the heterogeneity of the medium. As we show below, the finite-size effects are due to the fluctuation part. In the following section we investigate certain cases in which explicit solutions for the kernel are possible.

III. SOLUTIONS FOR THE EFFECTIVE PERMEABILITY KERNEL

First we investigate stochastically (macroscopically) isotropic media in which the two-point correlation $c_K(\mathbf{s}-\mathbf{s}')$ is a spherically symmetric function of the distance $r=|\mathbf{s}-\mathbf{s}'|$. This restriction does not limit the analysis, since anisotropic media can be reduced to isotropic by a rescaling of the coordinate axes as we show below. The Fourier transform of the isotropic dipolar tensor is given by

$$\widetilde{G}_{0ij}(\mathbf{k})=-\frac{k_ik_j}{k^2\overline{K}}, \tag{3.1}$$

and the effective permeability kernel is generally expressed as follows [23]:

$$\widetilde{K}_{ij}^{*}(k)=\widetilde{K}_l^{*}(k)\widetilde{P}_{ij}(\mathbf{k})+\widetilde{K}_t^{*}(k)\widetilde{R}_{ij}(\mathbf{k}), \tag{3.2}$$

where $\widetilde{K}_l^{*}(k)$ denotes the longitudinal and $\widetilde{K}_t^{*}(k)$ the transverse effective permeability components; $\widetilde{P}_{ij}(\mathbf{k})$ and $\widetilde{R}_{ij}(\mathbf{k})$ represent orthogonal, idempotent projection operators that in momentum space are given by

$$\widetilde{P}_{ij}(\mathbf{k})=\frac{k_ik_j}{k^2} \tag{3.3}$$

and

$$\widetilde{R}_{ij}(\mathbf{k})=\delta_{ij}-\widetilde{P}_{ij}(\mathbf{k}). \tag{3.4}$$

Equation (2.13) then becomes

$$\widetilde{M}_{ij}^{-1}(\mathbf{k})=\delta_{ij}+\frac{\overline{K}-\widetilde{K}_l^*(k)}{\overline{K}}\,\widetilde{P}_{ij}(\mathbf{k}). \tag{3.5}$$

Equation (3.5) can be inverted, using the idempotence of $\widetilde{P}_{ij}(\mathbf{k})$, to obtain

$$\widetilde{M}_{ij}(\mathbf{k})=\widetilde{R}_{ij}(\mathbf{k})+\widetilde{P}_{ij}(\mathbf{k})\,\frac{\overline{K}}{\widetilde{K}_l^*(k)}. \tag{3.6}$$

The two-point correlation $\widetilde{M}_{ij}(\mathbf{k})$ can be eliminated from Eq. (2.21) using Eq. (3.6) combined with the properties of the projectors. The following equation is thus obtained:

$$\widetilde{K}_{ij}^*(\mathbf{k})=\delta_{ij}\overline{K}-\int\frac{\mathbf{dk}_1}{(2\pi)^d}\,\widetilde{c}_K(\mathbf{k}-\mathbf{k}_1)\,\frac{\widetilde{P}_{ij}(\mathbf{k}_1)}{\widetilde{K}_l^*(\mathbf{k}_1)}. \tag{3.7}$$

Equation (3.7) can be solved exactly at the infrared limit $k\xi\to 0$, where we obtain

$$\widetilde{K}_{ij}^*(0)=\delta_{ij}\overline{K}-\frac{c_K(\mathbf{r}=0)}{\widetilde{K}_l^*(0)}\int\frac{d\Omega_d}{S_d}\,\widetilde{P}_{ij}(\Omega_d)$$
$$=\delta_{ij}\left(\overline{K}-\frac{\sigma_K^2}{d\widetilde{K}_l^*(0)}\right), \tag{3.8}$$

where $\int d\Omega_d$ denotes integration over the surface of the d-dimensional unit sphere. In view of the projection decomposition of the effective permeability kernel Eq. (3.8) leads to the following expressions for the longitudinal and transverse components

$$\widetilde{K}_l^*(0)=\overline{K}-\frac{\sigma_K^2}{d\widetilde{K}_l^*(0)} \tag{3.9}$$

and

$$\widetilde{K}_t^*(0)=\widetilde{K}_l^*(0), \tag{3.10}$$

where $\widetilde{K}_l^*(0)=\widetilde{K}_l^*(k=0)$ and $\widetilde{K}_t^*(0)=\widetilde{K}_t^*(k=0)$. The solution of Eq. (3.9) for $\widetilde{K}_l^*(0)$ is

$$\widetilde{K}_l^*(0)=\frac{\overline{K}}{2}\left[1+\left(1-\frac{4\sigma_K^2}{d\overline{K}^2}\right)^{1/2}\right]. \tag{3.11}$$

Equation (3.11) is the variational approximation obtained by means of the optimal Gaussian measure and is valid at all perturbation orders. Hence, we believe that it provides more accurate estimates than low-order perturbation calculations in the case of moderate heterogeneity. At the limit of weak heterogeneity $\mu\ll 1$ the longitudinal kernel $\widetilde{K}_l^*(0)$ is given by the first-order approximation

$$\widetilde{K}_l^{*(1)}(0)=\overline{K}-\frac{\sigma_K^2}{d\overline{K}}, \tag{3.12}$$

which agrees with standard first-order perturbation analysis [12].

In the case of a uniform mean pressure gradient $\overline{\mathbf{J}}$, the following uniform effective permeability is obtained after integration of the exact kernel

$$\mathbf{K}_{\rm eff}(\infty)=\int\mathbf{ds}'\,\mathbf{K}^*(s-s')=\widetilde{\mathbf{K}}^*(k=0). \tag{3.13}$$

This asymptotic $\mathbf{K}_{\rm eff}(\infty)$ is involved in the effective Darcy's law $\overline{\mathbf{Q}}=\mathbf{K}_{\rm eff}(\infty)\cdot\overline{\mathbf{J}}$, that is widely used in hydrology [1]. In view of Eqs. (3.2) and (3.10), $\mathbf{K}_{\rm eff}(\infty)$ can be replaced by the scalar permeability

$$K_{\rm eff}(\infty)=\widetilde{K}_l^*(0). \tag{3.14}$$

The variational approximation of $K_{\rm eff}(\infty)$ given by Eq. (3.11), satisfies the standard bounds

$$(\overline{K^{-1}})^{-1}\leqslant K_{\rm eff}(\infty)\leqslant\overline{K}. \tag{3.15}$$

In view of Eqs. (3.12) and (3.14) the first-order approximation of the effective permeability is given by

$$K_{\rm eff}^{(1)}=\overline{K}-\frac{\sigma_K^2}{d\overline{K}}. \tag{3.16}$$

It should be emphasized that the lowest disorder correction is fully determined by two-point correlations. Hence, it is a valid approximation even for non-Gaussian permeability distributions.

In the preasymptotic regime Eq. (3.7) leads to the following expressions for the longitudinal and transverse components:

$$\widetilde{K}_l^*(\mathbf{k})=\overline{K}-\int\frac{\mathbf{dk}_1}{(2\pi)^d}\,\frac{\widetilde{c}_K(\mathbf{k}-\mathbf{k}_1)}{\widetilde{K}_l^*(\mathbf{k}_1)}\,\frac{(\mathbf{k}\cdot\mathbf{k}_1)^2}{k^2k_1^2}, \tag{3.17}$$

and

$$\widetilde{K}_t^*(\mathbf{k})=\overline{K}-\frac{1}{d-1}\int\frac{\mathbf{dk}_1}{(2\pi)^d}\,\frac{\widetilde{c}_K(\mathbf{k}-\mathbf{k}_1)}{\widetilde{K}_l^*(\mathbf{k}_1)}\left[1-\frac{(\mathbf{k}\cdot\mathbf{k}_1)^2}{k^2k_1^2}\right]. \tag{3.18}$$

Note that Eq. (3.17) is a nonlinear integral equation in $\widetilde{K}_l^*(\mathbf{k})$, while $\widetilde{K}_t^*(\mathbf{k})$ is given by means of a momentum integral of the $\widetilde{K}_l^*(\mathbf{k})$. The longitudinal (transverse) permeability can be decomposed into a mean part equal to $\overline{K}$ and a fluctuation part $\delta\widetilde{K}_l^*(\mathbf{k})$ [$\delta\widetilde{K}_t^*(\mathbf{k})$]. The kernel $\delta\widetilde{K}_l^*(\mathbf{k})$ is given by means of the following equation:

$$\delta\widetilde{K}_l^*(\mathbf{k})=\int\frac{\mathbf{dk}_1}{(2\pi)^d}\,\frac{\widetilde{c}_K(\mathbf{k}-\mathbf{k}_1)}{\overline{K}-\delta\widetilde{K}_l^*(\mathbf{k}_1)}\,\frac{(\mathbf{k}\cdot\mathbf{k}_1)^2}{k^2k_1^2}. \tag{3.19}$$

The ultraviolet limit of $\delta\widetilde{K}_l^*(\mathbf{k})$ is obtained from Eq. (3.19) by means of the transformation $\mathbf{k}'=\mathbf{k}-\mathbf{k}_1$ that leads to the following expression:

$$\delta\widetilde{K}_l^*(\infty)=\frac{\sigma_K^2}{\overline{K}-\delta\widetilde{K}_l^*(\infty)}. \tag{3.20}$$

Using Eq. (3.20) the longitudinal kernel $\widetilde{K}_l^*(\mathbf{k})$ can be expressed as

$$\widetilde{K}_l^*(\mathbf{k})=\overline{K}-\delta\widetilde{K}_l^*(\infty)+\delta\widetilde{K}_{l;\mathrm{nloc}}^*(\mathbf{k}), \tag{3.21}$$

where

$$\delta\widetilde{K}_{l;\mathrm{nloc}}^*(\mathbf{k})=\delta\widetilde{K}_l^*(\infty)-\delta\widetilde{K}_l^*(\mathbf{k}). \tag{3.22}$$

The uniform part in $\widetilde{K}_l^*(\mathbf{k})$ represents the local component that leads to a δ function kernel in real space. The $\delta\widetilde{K}_{l;\mathrm{nloc}}^*(\mathbf{k})$ represents the nonlocal component of the kernel. Analogous relations hold for the transverse kernel $\widetilde{K}_t^*(\mathbf{k})$. Hence, the fluctuation kernel can be decomposed into local and nonlocal components as follows:

$$\delta\widetilde{K}_{ij}^*(\mathbf{k})=[\delta\widetilde{K}_l^*(\infty)\widetilde{P}_{ij}(\mathbf{k})+\delta\widetilde{K}_l^*(\infty)\widetilde{R}_{ij}(\mathbf{k})]-[\delta\widetilde{K}_{l;\mathrm{nloc}}^*(k)\widetilde{P}_{ij}(\mathbf{k})+\delta\widetilde{K}_{t;\mathrm{nloc}}^*(k)\widetilde{R}_{ij}(\mathbf{k})]. \tag{3.23}$$

Finally, we evaluate the effective kernel in the case of a stratified, macroscopically anisotropic, three-dimensional medium with correlation length $\xi_1=\xi_2=\xi_\parallel$ in the longitudinal (in-plane) direction and $\xi_3=\xi_\perp=e\xi_\parallel$ in the transverse (out-of-plane) direction, where e denotes the anisotropy ratio. In this case, Eq. (3.7) is valid in the rescaled coordinate system $k_i'=\xi_i k_i$ in which the medium looks isotropic. The following expression is obtained:

$$\widetilde{K}_{ij}^*(0)=\delta_{ij}\overline{K}-\frac{\sigma_K^2}{\widetilde{K}_l^*(0)}\,\vartheta_{ij}, \tag{3.24}$$

where ϑ_{ij} is the anisotropy tensor

$$\vartheta_{ij}=\frac{\xi_i\xi_j}{\xi_\parallel^2}\int\frac{d\Omega}{4\pi}\,\frac{\theta_i\theta_j}{(\sin^2\chi+e^2\cos^2\chi)}, \tag{3.25}$$

and θ_i denotes the direction cosine. The elements of the diagonal anisotropy tensor are given by

$$\vartheta_{11}=\vartheta_{22}=\frac{1}{2(e^2-1)}\left(\frac{e^2\tan^{-1}\sqrt{e^2-1}}{\sqrt{e^2-1}}-1\right), \tag{3.26}$$

$$\vartheta_{33}=\frac{e^2}{e^2-1}\left(1-\frac{\tan^{-1}\sqrt{e^2-1}}{\sqrt{e^2-1}}\right), \tag{3.27}$$

and they are plotted in Fig. 1 versus the anisotropy ratio. The effect of the fluctuations is more pronounced, leading to lower effective permeability, in the direction of higher correlation.

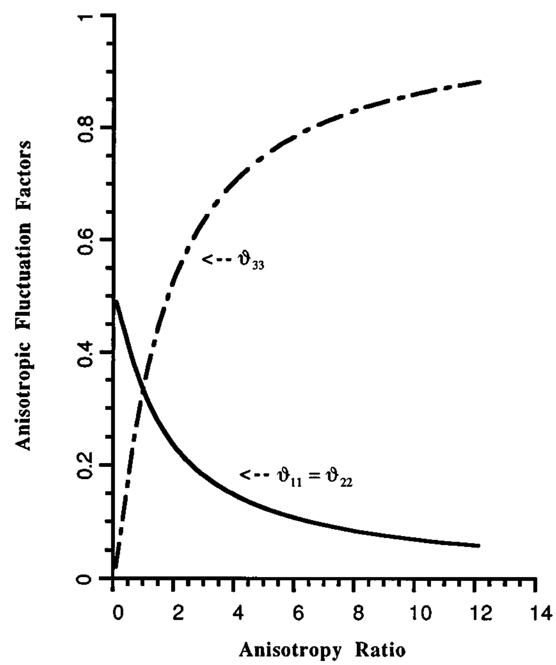

FIG. 1. Plot of the longitudinal and transverse components of the anisotropy tensor vs the anisotropy ratio.

IV. PERTURBATION ANALYSIS AND SOLUTION OF THE DISPERSIVE PART

The main focus of the analysis is the solution of the nonlinear integral equation (3.17) that determines the longitudinal effective permeability kernel. Equation (3.17) can be classified as a nonlinear Fredholm equation of the second kind that can be solved numerically [24,25]. A solution can be obtained by means of the iterative method [26] starting with $\widetilde{K}_{l(0)}^*(\mathbf{k})=\overline{K}$ as the initial approximation. A different approach is by means of a perturbation expansion [25] that leads to a power series in the permeability variance. The partial sum of all orders lower than or equal to n is denoted by $\widetilde{K}_l^{*(n)}(k)$. Since Eq. (3.17) is nonlinear $\widetilde{K}_{l(n)}^*(\mathbf{k})$ is in general different from $\widetilde{K}_l^{*(n)}(k)$, with the exception of $n=0$ and $n=1$. Both methods should converge for weak disorder $\mu<1$. Let $D(k)$ denote the dispersive part of the effective permeability kernel that represents the following infinite sum:

$$D(k)=\sum_{n=1}^{\infty}g_n(k)\mu^{2n}. \tag{4.1}$$

In view of Eq. (4.1) $\widetilde{K}_l^*(k)=\overline{K}[1-D(k)]$, and Eq. (3.8) is then expressed as

$$D(k)=\mu^2\int\frac{\mathbf{dk}_1}{(2\pi)^d}\,\frac{\widetilde{\rho}_K(\mathbf{k}-\mathbf{k}_1)}{1-D(k_1)}\,\frac{(\mathbf{k}\cdot\mathbf{k}_1)^2}{k^2k_1^2}, \tag{4.2}$$

where $\widetilde{\rho}_K(\mathbf{k})$ denotes the Fourier transform of the permeability correlation function. The functions $g_n(k)$ are determined by equating the coefficients of equal powers in μ on both sides of Eq. (4.2). The first dispersion function is

$$g_1(k)=\int d\Omega_d\int_0^\infty\frac{dk_1}{(2\pi)^d}\,\frac{k_1^{d-1}(k-\hat{\mathbf{k}}\cdot\mathbf{k}_1)^2}{(\mathbf{k}-\mathbf{k}_1)^2}\,\widetilde{\rho}_K(k_1), \tag{4.3}$$

where $\hat{\mathbf{k}}$ is the unit vector in the direction of $\mathbf{k}$. It can be shown that the $g_n(k)$ are positive everywhere; thus, the convergence criterion for the series is $\mu^2 g_{n+1}(k)\leqslant g_n(k)$ for all $n\geqslant 1$, and the error incurred by terminating the series at or-

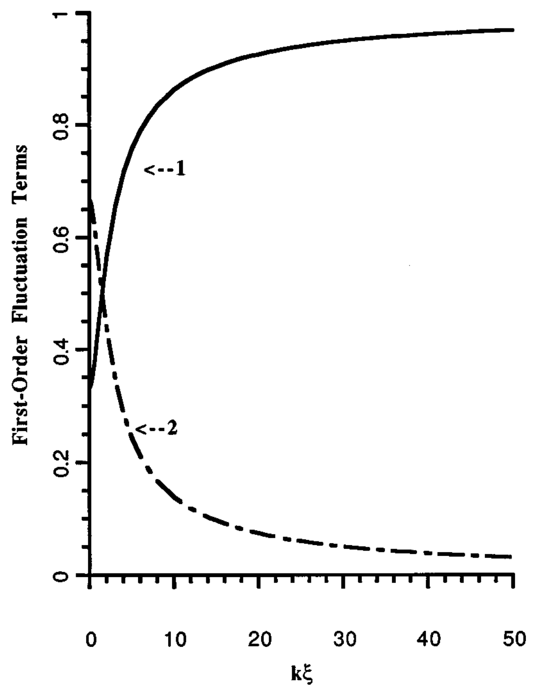

FIG. 2. Plots of the dispersive part $g_1(k)$ (curve 1) and the nonlocal component $g_1(\infty)-g_1(k\xi)$ (curve 2) of the effective permeability vs the dimensionless momentum $k\xi$. Both curves have been obtained for an exponential permeability correlation model with length ξ.

der n is always less than $g_{n+1}(k)\mu^{2n+1}$ [27]. Hence, the effective permeability kernel can be approximated with arbitrary accuracy in the case $\mu<1$.

We assume a weakly heterogeneous medium with $\mu^2 \ll 1$. In the case of a one-dimensional medium we obtain the following expression for the perturbation $g_1(k)$:

$$g_1(k)=\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\,\tilde{\rho}(k_1)=1. \tag{4.4}$$

Hence, for $d=1$ the effective permeability kernel is purely local. Next, we assume an isotropic medium with an exponential two-point permeability correlation function

$$\rho_K(\mathbf{r})=\exp\left(-\frac{r}{\xi}\right), \tag{4.5}$$

where ξ is the isotropic permeability correlation length. The Fourier transform of $\rho_K(\mathbf{r})$ is given by

$$\tilde{\rho}_K(k)=\frac{2^d\xi^d\pi^{d/2}}{\Gamma(2-d/2)}\,\frac{1}{(1+\xi^2k^2)(d+1)/2}, \tag{4.6}$$

where $\Gamma(\,)$ denotes the Γ function. The function $g_1(k)$ is evaluated in the Appendix for $d=3$, and it is plotted in Fig. 2 (curve 1) versus the dimensionless momentum $k\xi$. The function $g_1(k)$ and consequently $\delta\tilde{K}_l^{*(1)}(k)$ increase monotonically with $k\xi$. Both functions have a broader Fourier spectrum than the two-point correlation (4.6), which decays to zero within the range $k\xi\cong 1$. This behavior is due to the convolution of the permeability two-point correlation with the projection operator **P**. In the Appendix we also obtain the infrared limit

$$\lim_{k\xi\to 0}\delta\tilde{K}_l^{*(1)}(k)=\frac{\sigma_K^2}{3\bar{K}}, \tag{4.7}$$

which is in agreement with Eq. (3.12), and the ultraviolet limit

$$\lim_{k\xi\to\infty}\delta\tilde{K}_l^{*(1)}(k)=\frac{\sigma_K^2}{\bar{K}}. \tag{4.8}$$

The nonlocal part of the permeability kernel perturbation is given by

$$\delta\tilde{K}_{l;\mathrm{nloc}}^{*(1)}(k)=\frac{\sigma_K^2}{\bar{K}}[g_1(\infty)-g_1(k\xi)]. \tag{4.9}$$

The function $\delta\tilde{K}_{l;\mathrm{nloc}}^{*(1)}(k)$ is a positive and monotonically decreasing function of the dimensionless momentum $k\xi$, which tends to zero at infinity. In Fig. 2 (curve 2) we plot the nonlocal component $g_1(\infty)-g_1(k\xi)$ versus $k\xi$.

V. EFFECTIVE PERMEABILITY OF FINITE MEDIUM

The analysis of the effective permeability kernel equations in Secs. III and IV above is based on the assumption of a medium with an infinite support scale, which is in agreement with a scale-independent effective permeability. This assumption is justified only if the size of the support significantly exceeds the memory length of the effective kernel. In the case of a medium with finite support, the effective permeability depends on the boundaries of the medium, namely, on the geometry of the support and the imposed boundary conditions on the pressure field. In the following, it is assumed that an isotropic length scale L characterizes the size of the support. The effective permeability should reflect changes in the support scale due to the cutoff of correlations between points separated by a distance greater than allowed by the support size. A detailed analysis of the effective permeability of finite media should employ the homogeneous Green's function that honors the boundary conditions. However, the Green's function in a bounded domain is usually obtained in terms of an infinite series, e.g., by means of an eigenfunction expansion, and it lacks translational invariance. This complicates the mathematical analysis so that even low-order calculations become very tedious. Here, we present an approximate model for the finite-size effects that circumvents these difficulties; this model can be viewed as an estimator that interpolates the effective permeability between the arithmetic mean and the asymptotic value. We assume Dirichlet pressure boundaries in the longitudinal direction and Neumann no-flow boundaries in the transverse directions. We use the infinite Green's function and thus neglect finite-size effects near the boundaries. This assumption permits the use of the translationally invariant equations for the effective permeability obtained in Secs. II–IV above. We model the effect of the support on the effective permeability by means of fluctuation filters that cut off the spectral density outside a sphere with radius equal to $k_{\max}=\Lambda=2\pi/L$ in momentum space. The mean flux at location $\mathbf{s}$ is then given by the following filtered equation:

$$\overline{\mathbf{Q}}_{\Lambda}(\mathbf{s})=\int \mathbf{ds}'\,\Phi_{\Lambda}(\mathbf{s}-\mathbf{s}')[\mathbf{K}^{*}_{\Lambda}(\mathbf{s}-\mathbf{s}')\cdot\overline{\mathbf{J}}(\mathbf{s}')], \quad (5.1)$$

where the convolution integral is unconstrained, but the filter function $\Phi_{\Lambda}(\mathbf{s}-\mathbf{s}')$ cuts off the integral when the point $\mathbf{s}'$ is located outside the support boundaries. The nonlocal effective permeability $\mathbf{K}^{*}_{\Lambda}(\mathbf{s}-\mathbf{s}')$ also has an implicit scale dependence, which we investigate in more detail below. The effect of the filter depends on the size of the memory length of the kernel $\mathbf{K}^{*}_{\Lambda}(\mathbf{s}-\mathbf{s}')$ relative to the support scale: if the memory length is significantly smaller than the support scale, the filter can be eliminated from Eq. (5.1) leading to a mean flux independent of the support size. In the case of a uniform mean pressure gradient Eq. (5.1) can be expressed as

$$\overline{\mathbf{Q}}(\Lambda)=\mathbf{K}_{\rm eff}(\Lambda)\overline{\mathbf{J}}, \quad (5.2)$$

where the scale-dependent uniform effective permeability is given by

$$\mathbf{K}_{\rm eff}(\Lambda)=\overline{K}\mathbf{I}-\int \frac{\mathbf{dk}}{(2\pi)^{d}}\,\tilde{\Phi}_{\Lambda}(k)\,\boldsymbol{\delta}\tilde{\mathbf{K}}^{*}_{\Lambda}(\mathbf{k}), \quad (5.3)$$

and the fluctuation permeability component is

$$\boldsymbol{\delta}\mathbf{K}_{\rm eff}(\Lambda)=\int \frac{\mathbf{dk}}{(2\pi)^{d}}\,\tilde{\Phi}_{\Lambda}(k)\,\boldsymbol{\delta}\tilde{\mathbf{K}}^{*}_{\Lambda}(\mathbf{k}). \quad (5.4)$$

Let $V_d(\Lambda)$ denote the volume of the d-dimensional sphere and $\theta(\Lambda-k)$ the unit step function. The filter function

$$\tilde{\Phi}_{\Lambda}(k)=\frac{(2\pi)^{d}}{V_d(\Lambda)}\,\theta(\Lambda-k) \quad (5.5)$$

eliminates $\boldsymbol{\delta}\mathbf{K}_{\rm eff}(\Lambda)$ for a pointlike support, while it yields the asymptotic effective permeability for an infinite support. The integral equations that determine the effective permeability kernels are also filtered with an appropriate function $\tilde{\Psi}_{\Lambda}(k)$ that eliminates the fluctuations in the case of vanishing support size and yields the nonfiltered equations, namely (3.17) and (3.18), in the case of infinite support scale. These constraints are satisfied by the momentum filter

$$\tilde{\Psi}_{\Lambda}(k)=\theta(k-\Lambda). \quad (5.6)$$

In the case of macroscopic isotropy the fluctuation component $\boldsymbol{\delta}\mathbf{K}_{\rm eff}(\Lambda)$ of Eq. (5.4) is expressed as

$$\boldsymbol{\delta}\mathbf{K}_{\rm eff}(\Lambda)=\frac{1}{(2\pi)^{d}}\int_{0}^{\infty}dk\;k^{d-1}\tilde{\Phi}_{\Lambda}(k)\left(\int d\Omega_d\,\boldsymbol{\delta}\tilde{\mathbf{K}}^{*}_{\Lambda}(\mathbf{k})\right), \quad (5.7)$$

where $\boldsymbol{\delta}\tilde{\mathbf{K}}^{*}_{\Lambda}(\mathbf{k})$ is a tensor with isotropic momentum dependence. Evaluation of the surface integral leads to an isodiagonal permeability tensor following the equation

$$\int d\Omega_d\,\boldsymbol{\delta}\tilde{\mathbf{K}}^{*}_{\Lambda}(k)=\mathbf{I}\Omega_d\tilde{\mathcal{I}}_K(k;\Lambda), \quad (5.8)$$

where $\tilde{\mathcal{I}}_K(k;\Lambda)$ is a functional of the permeability distribution that represents the spectral density of $\boldsymbol{\delta}\mathbf{K}_{\rm eff}(\Lambda)$ and satisfies the equation

$$\tilde{\mathcal{I}}_K(k;\Lambda)=\frac{\mathrm{tr}\;\boldsymbol{\delta}\tilde{\mathbf{K}}^{*}_{\Lambda}(\mathbf{k})}{d}. \quad (5.9)$$

In view of Eqs. (5.3), (5.4), and (5.7)–(5.9) the scalar permeability is given by

$$K_{\rm eff}(\Lambda)=\overline{K}-\delta K_{\rm eff}(\Lambda), \quad (5.10)$$

where $K_{\rm eff}(\Lambda)$ and $\delta K_{\rm eff}(\Lambda)$ represent the magnitudes of the diagonal kernels $\mathbf{K}_{\rm eff}(\Lambda)$ and $\boldsymbol{\delta}\mathbf{K}_{\rm eff}(\Lambda)$. The fluctuation component $\delta K_{\rm eff}(\Lambda)$ is given by

$$\delta K_{\rm eff}(\Lambda)=\int \frac{\mathbf{dk}}{(2\pi)^{d}}\,\tilde{\Phi}_{\Lambda}(k)\tilde{\mathcal{I}}_K(k;\Lambda). \quad (5.11)$$

The asymptotic effective permeability $\delta K_{\rm eff}(0)$ is given in terms of the spectral density function $\tilde{\mathcal{I}}_K(k;\Lambda)$ by means of the expression

$$\delta K_{\rm eff}(\Lambda=0)=\tilde{\mathcal{I}}_K(k=0;\Lambda=0). \quad (5.12)$$

The spectral density $\tilde{\mathcal{I}}_K(k;\Lambda)$ can be evaluated using Eqs. (5.9) and (3.17) that lead to the following:

$$\tilde{\mathcal{I}}_K(k;\Lambda)=\frac{1}{d}\int \frac{\mathbf{dk}_1}{(2\pi)^{d}}\,\frac{\tilde{c}_K(\mathbf{k}-\mathbf{k}_1)}{\tilde{K}^{*}_{l}(k_1)}\,\tilde{\Psi}_{\Lambda}(k_1). \quad (5.13)$$

In light of Eq. (4.2) $\tilde{\mathcal{I}}_K(k;\Lambda)$ is given by means of the following expression:

$$\tilde{\mathcal{I}}_K(k;\Lambda)=\frac{\sigma_K^2}{d\overline{K}}\int \frac{\mathbf{dk}_1}{(2\pi)^{d}}\,\frac{\tilde{\rho}_K(\mathbf{k}-\mathbf{k}_1)\tilde{\Psi}_{\Lambda}(k_1)}{1-D(k_1)}. \quad (5.14)$$

The first-order approximation of the spectral density is

$$\tilde{\mathcal{I}}^{(1)}_K(k;\Lambda)=\frac{\sigma_K^2}{d\overline{K}}\tilde{f}^{(1)}(k;\Lambda), \quad (5.15)$$

where $\tilde{f}^{(1)}(k;\Lambda)$ denotes the normalized spectral density that represents the following integral

$$\tilde{f}^{(1)}(k;\Lambda)=\int \frac{\mathbf{dk}_1}{(2\pi)^{d}}\,\tilde{\rho}_K(\mathbf{k}-\mathbf{k}_1)\tilde{\Psi}_{\Lambda}(k_1). \quad (5.16)$$

Thus, the first-order fluctuation perturbation of $\delta K_{\rm eff}(\Lambda)$ is given by

$$\delta K^{(1)}_{\rm eff}(\Lambda)=\frac{\sigma_K^2}{d\overline{K}}h^{(1)}(\Lambda), \quad (5.17)$$

where the support function $h^{(1)}(\Lambda)$ represents the following integral

$$h^{(1)}(\Lambda)=\frac{d}{\Lambda^{d}}\int_{0}^{\Lambda}dk\;k^{d-1}\tilde{f}^{(1)}(k;\Lambda). \quad (5.18)$$

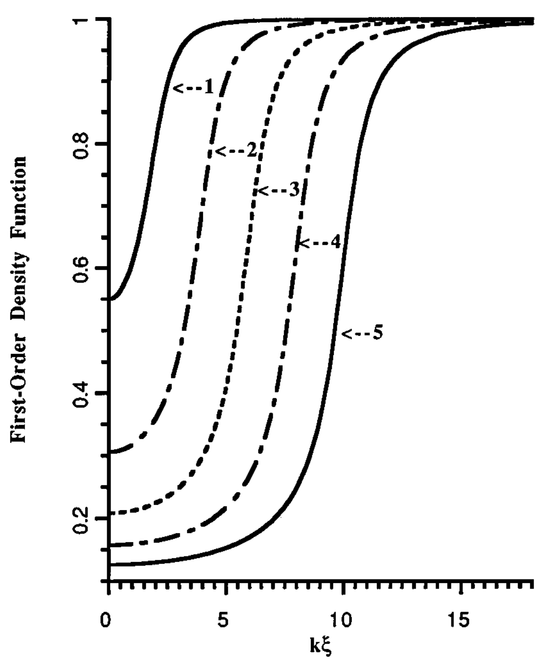

FIG. 3. The first-order spectral density function $\widetilde{f}^{(1)}(k;\Lambda)$ vs the dimensionless momentum $k\xi$ for an exponential permeability correlation model. Five different values of the dimensionless cutoff $\Lambda\xi$ are shown: $\Lambda\xi=2$ (curve 1), $\Lambda\xi=4$ (curve 2), $\Lambda\xi=6$ (curve 3), $\Lambda\xi=8$ (curve 4), and $\Lambda\xi=10$ (curve 5).

In the case of a three-dimensional medium with exponential decay of permeability correlations, the functions $\widetilde{f}^{(1)}(k;\Lambda)$ and $h^{(1)}(\Lambda)$ are given by the following expressions:

$$\widetilde{f}^{(1)}(k;\Lambda)=1-\frac{1}{2k\xi\pi}\ln\left[\frac{1+\xi^2(k-\Lambda)^2}{1+\xi^2(k+\Lambda)^2}\right]-\frac{1}{\pi}\tan^{-1}\left[\frac{2\xi\Lambda}{1+\xi^2(k^2-\Lambda^2)}\right] \tag{5.19}$$

and

$$h^{(1)}(\Lambda)=\frac{1}{2}-\frac{1}{\pi\Lambda\xi}-\frac{1}{\pi}\tan^{-1}\left(\frac{4\xi^2\Lambda^2-1}{4\Lambda\xi}\right)+\left(\frac{1+6\xi^2\Lambda^2}{4\pi\xi^3\Lambda^3}\right)\ln(4\xi^2\Lambda^2+1). \tag{5.20}$$

In Fig. 3 we plot the normalized spectral density $\widetilde{f}^{(1)}(k;\Lambda)$ versus the dimensionless momentum $k\xi$ for different values of the cutoff $\Lambda\xi$. The normalized spectral density increases monotonically and tends asymptotically to unity. In Fig. 4 we plot the $\widetilde{f}^{(1)}(k;\Lambda)$ versus the cutoff $\Lambda\xi$ for three different values of the momentum $k\xi$. The $\widetilde{f}^{(1)}(k;\Lambda)$ is a decreasing function of the cutoff that tends asymptotically to zero when $\Lambda\xi$ tends to infinity. In Fig. 5 we present a plot of the support function $h^{(1)}(\Lambda)$ versus the cutoff. The support function decreases (increases) monotonically with the dimensionless cutoff (support scale) from one (zero) to zero (one) with a rate that is slower than exponential. This becomes more evident in terms of the function

$$\eta^{(1)}\left(\frac{L}{\xi}\right)=-\ln[h^{(1)}(\Lambda)-1], \tag{5.21}$$

which we plot in Fig. 6 versus the dimensionless support scale. If the support function decreased exponentially, the plot of $\eta^{(1)}(L/\xi)$ would be a straight line with positive slope.

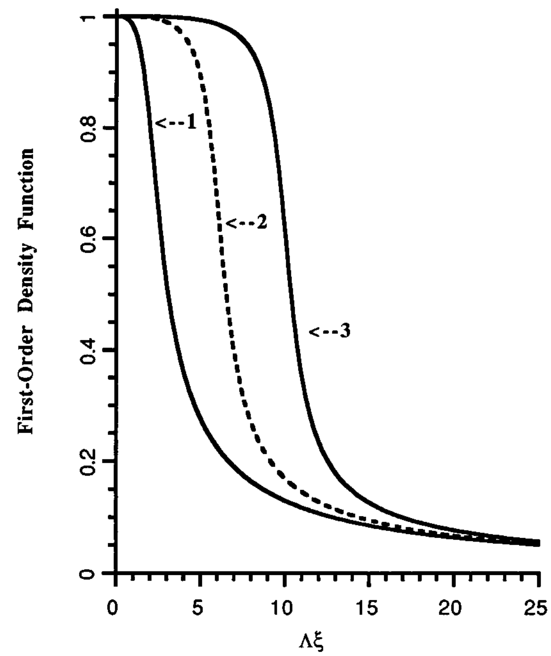

FIG. 4. The first-order spectral density function $\widetilde{f}^{(1)}(k;\Lambda)$ vs the dimensionless cutoff $\Lambda\xi$ for an exponential permeability correlation model. Three different values of the dimensionless momentum are shown: $k\xi=2$ (curve 1), $k\xi=6$ (curve 2), and $k\xi=10$ (curve 3).

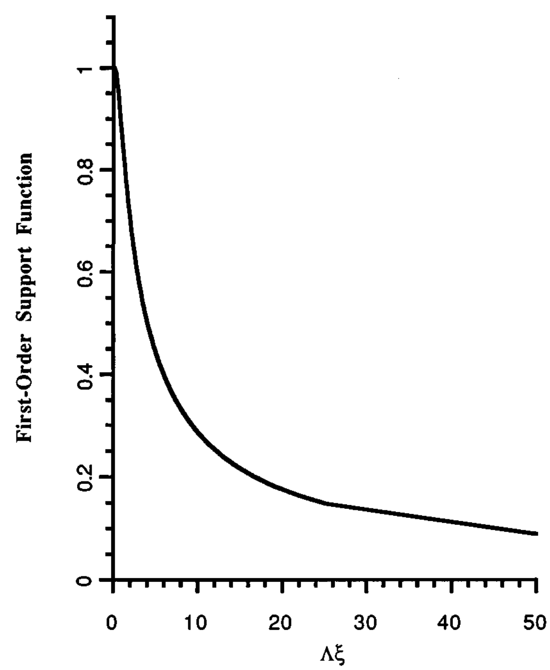

FIG. 5. Plot of the support function $h^{(1)}(\Lambda)$ vs the dimensionless momentum cutoff $\Lambda\xi$.

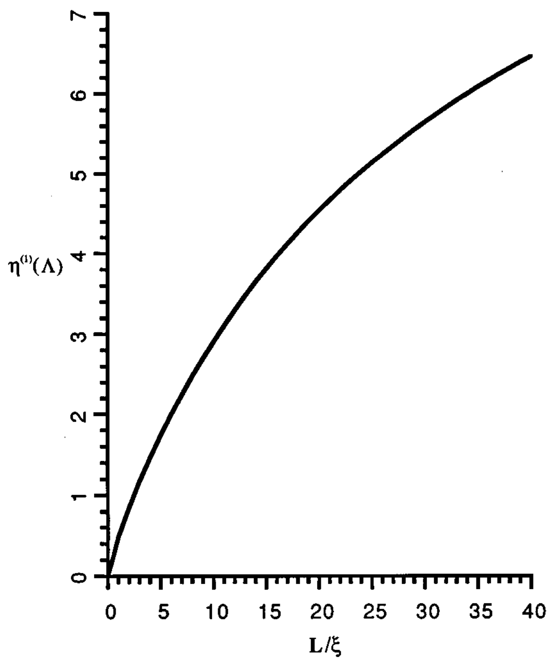

FIG. 6. Plot of the function $\eta^{(1)}(\Lambda)$ vs the dimensionless support scale L/ξ.

Instead, $\eta^{(1)}(L/\xi)$ is a convex up function, which indicates a slower-than-exponential decay.

Finally, the preasymptotic uniform effective permeability is given by

$$K_{\mathrm{eff}}^{(1)}(\Lambda)=\overline{K}-\frac{\sigma_K^2}{3\overline{K}}h^{(1)}(\Lambda), \qquad (5.22)$$

The finite-size behavior represented by Eq. (5.22) is physically meaningful: at the ultraviolet limit $\Lambda \to \infty$, i.e., for vanishing support length, Eq. (5.22) tends to the arithmetic mean, while at the infrared limit $\Lambda \to 0$, i.e., for an infinite support, Eq. (5.22) tends to the value of the asymptotic effective permeability.

VI. COMPARISONS WITH EXPERIMENTAL RESULTS

In this section we compare our theoretical analysis with experimental and numerical estimates of the effective permeability obtained in [18]. We begin by pointing out a systematic difference: in our work the effective permeability follows from a stochastic average over the local permeability probability distribution. In the approach of [18] the effective permeability is evaluated either experimentally or numerically using spatial averaging of the fluid flux. As we mentioned in the Introduction the spatial and the ensemble averages coincide only if the flux and the pressure gradient are ergodic. The authors in [18] have obtained permeability measurements for two porous blocks of sandstone and limestone of dimensions $15\times 15\times 50$ cm^3. Each of the original samples was cut into three intermediate blocks of dimensions 15^3 cm^3, and finally 25 cylindrical plugs were obtained from each intermediate block. The local permeability probability distribution was inferred from the 300 samples obtained from each block. The effective permeabilities for mixed Dirichlet-Neumann conditions were measured at the three scales. The effective permeability of the intermediate blocks was also evaluated by solving Darcy's equation numerically and using the box integration method as explained in [18]. Numerical simulations were carried out on 20 random samples that were generated based on the experimental probability distributions, and the permeability mean was evaluated numerically and compared to the experimental measurements. The coefficient of variation for the sandstone blocks was found to satisfy $\mu < 1$, while for the limestone blocks it was found that $\mu > 1$. Our low-order explicit calculations are applicable only in the case of the sandstone experiments. For $\mu > 1$ estimates that account for the large distribution variance are required, e.g., [9,13,28].

In Table I we compare the effective permeability obtained using the first-order perturbation analysis above with the experimental and numerical values for the sandstone obtained in [18]. The correlation lengths of the local permeability were estimated in [18] to be about 2% of the support length, in view of which we use the asymptotic expression (3.16) for the $K_{\mathrm{eff}}^{(1)}$. The first four columns of Table I are taken from Table 5 of [18]: they represent from left to right the arithmetic mean, the coefficient of variation, the effective permeability $K_{\mathrm{eff}}^{(\mathrm{meas})}$ measured in the laboratory, and the calculated effective permeability $K_{\mathrm{eff}}^{(\mathrm{cal})}$. The fifth column is the perturbation estimate obtained by means of Eq. (3.16). Note that the third intermediate block is characterized by a measured value that exceeds the arithmetic mean. This was explained in [18] as being due to a large number of missing permeability data from the third block due to mechanical failures during the preparation of the samples. It is clear that $K_{\mathrm{eff}}^{(1)}$ approximates $K_{\mathrm{eff}}^{(\mathrm{meas})}$ more accurately than $K_{\mathrm{eff}}^{(\mathrm{cal})}$. However, both the $K_{\mathrm{eff}}^{(1)}$ and the $K_{\mathrm{eff}}^{(\mathrm{cal})}$ overestimate the experimental value $K_{\mathrm{eff}}^{(\mathrm{meas})}$. This systematic difference could be due to

TABLE I. The entries of this table provide a comparison between the laboratory values (column 3) and the numerical estimates (column 4) of the permeabilities in [3] and the variational approximation (column 5) of the effective permeability given by (3.16). All permeability values are in millidarcy (1 darcy $=0.987\times 10^{-12}$ m^2). The asterisk denotes that the measured permeability of the third block fails to satisfy the upper bound of Eq. (3.15).

Sandstone	$\overline{K}$	μ	$K_{\mathrm{eff}}^{(\mathrm{meas})}$	$K_{\mathrm{eff}}^{(\mathrm{cal})}$	$K_{\mathrm{eff}}^{(1)}$
First Intermediate block	311	0.35	270	304	298
Second Intermediate block	250	0.46	220	243	232
* Third Intermediate block	129	0.54	150	122	116
Global block	230	0.54	210	218	208

anisotropic effects. Finally, the effective permeability is seen to decrease from the value 230 mD, which is the total arithmetic mean at the local scale to 208 mD at the global scale. This is in general agreement with the behavior of the support function. However, a comprehensive test of the support function would require permeability measurements for several intermediate scales and accurate characterization of the correlation lengths.

VII. CONCLUSIONS

We have calculated the preasymptotic effective permeability of heterogeneous media using the replica-variational approach. We have given explicit solutions for the integral equations satisfied by the effective permeability kernel in specific cases, and we have evaluated the uniform effective permeability for homogeneous disorder. The effects of the support scale have been analyzed in terms of an approximate model that uses momentum filtering of the fluctuations. The finite-size behavior is obtained in terms of a support function that has been calculated to lowest order for exponential decay of correlations. The support function has been found to increase towards its asymptotic value at a rate that is slower than exponential. Higher-order solutions of the integral equations that determine the effective permeability kernel merit further study, possibly by means of numerical techniques. Finally, it has been assumed in this investigation that the disorder is homogeneous with short-range correlations. These assumptions do not adequately represent all natural porous media [15,29], even though they are sufficient for certain media at the laboratory scale.

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APPENDIX

In this Appendix we evaluate the integral of Eq. (4.3) for the exponential correlation model. The function $g_1(k)$ can be expressed as

$$g_1(k)=\frac{2\xi^3}{\pi}\int_{-1}^{1}dx\int_0^\infty dk_1\,\frac{k_1^2(k^2+k_1^2x^2-2kk_1x)}{(k^2+k_1^2-2kk_1x)(1+\xi^2k_1^2)^2}. \tag{A1}$$

Let

$$I_2^1(k,k_1)=\int_{-1}^{1}dx\,\frac{(k^2+k_1^2x^2-2kk_1x)}{(k^2+k_1^2-2kk_1x)}. \tag{A2}$$

The surface integral is readily performed using a partial fraction expansion of the integrand leading to

$$I_2^1(k,k_1)=\frac{3-(k_1/k)^2}{2}+\frac{1}{8}\ln\left(\frac{k+k_1}{k-k_1}\right)^2\frac{(k^2-k_1^2)^2}{k^3k_1}. \tag{A3}$$

The integral over the scalar momentum is evaluated numerically using a MAPLE integration routine based on an adaptive Newton-Cotes algorithm [30]. In order to obtain exact expressions of the low and high momentum limits we use in the regime $k\xi\leqslant 1$ a Taylor series expansion of the integrand around $k=0$, and we evaluate $g_1(k)$ by applying the residue theorem on the truncated integrand series. The first two terms of the series expansion of $g_1(k)$ are

$$g_1(k)=\frac{1}{3}+\frac{2k^2\xi^2}{15}+O(k^4\xi^4). \tag{A4}$$

At the limit $k\xi\rightarrow\infty$ we assume an asymptotic expansion of the integrand around $1/k=0$ and integrate using the residue theorem. The first two terms of this expansion are given by

$$g_1(k)=1+\frac{2}{k^2\xi^2}+O\left(\frac{1}{k^4\xi^4}\right). \tag{A5}$$

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